Int. J. of Applied Mechanics and Engineering, 2021, vol.26, No.2, pp.160-172
DOI: 10.2478/ijame-2021-0025

NUMERICAL SOLUTION OF SINGULARLY PERTURBED TWO PARAMETER PROBLEMS USING EXPONENTIAL SPLINES

P. PADMAJA
Department of Mathematics, Prasad V Potluri Siddhartha Institute of Technology
Vijayawada-520007, Andhra Pradesh, INDIA

P.APARNA
Department of Mathematics, Vallurupalli Nageswara Rao Vignana Jyothi Institute of Engineering and Technology (VNR VJIET) Bachupaly, Hyderabad-500090, INDIA

Rama Subba Reddy GORLA*
Department of Aeronautics and Astronautics, Air Force Institute of Technology
Wright Patterson Air Force Base Dayton, Ohio 45433, USA
E-mail: rama.gorla@afit.edu

N.POTHANNA
Department of Mathematics, Vallurupalli Nageswara Rao Vignana Jyothi Institute of Engineering and Technology (VNR VJIET) Bachupaly, Hyderabad-500090, INDIA

In this paper, we have studied a method based on exponential splines for numerical solution of singularly perturbed two parameter boundary value problems. The boundary value problem is solved on a Shishkin mesh by using exponential splines. Numerical results are tabulated for different values of the perturbation parameters. From the numerical results, it is found that the method approximates the exact solution very well.

Keywords: singular perturbation, convection-diffusion problem, exponential splines, numerical convergence.

1. Introduction

Singular perturbation problems occur in the theory of viscous flow, in certain problems in the theory of elasticity, and in many other branches of applied mathematics like in fluid flow at high Reynolds number [1, 2], simulation of oil extraction from underground reservoirs [3], convective heat transport problems [4], water quality problems in river networks [5], the driftdiffusion equations of semiconductor device physics [6, 7], the Michaelis-Menten theory for enzyme reactions [8], mathematical theory of liquid crystal materials and chemical reactions [9], etc. Two-dimensional singularly perturbed convection-diffusion equations are regarded as the simplest version of the Navier-Stokes equations [10]. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. It is well known that classical finite difference schemes cannot give satisfactory numerical results for singularly perturbed problems when the mesh size is greater than the perturbation parameter and in the case of mesh size less than the perturbation parameter, this will lead to a huge linear system. To overcome this, there are two types of methods: (i) fitted operator methods and (ii) fitted mesh methods. In the fitted operator methods, an appropriate fitting factor is to be used to find the numerical solution on uniform meshes. Stable numerical methods on uniform mesh for singularly perturbed ordinary differential equations are equivalent to adding sufficient artificial diffusion to the

* To whom correspondence should be addressed

differential equation before applying standard numerical method for approximation. First such method was proposed by Allen and Southwell [11] and its convergence in one dimension was analyzed by Il'in [12]. The amount of artificial diffusion is calculated such that the computed solution agrees with the exact solution at the mesh points. One of the landmark work in this direction is by Kellogg and Tsan [13], where the authors used comparison principle to deduce discretization error from the bounds on truncation error. This is one of the most used results now. El-Mistikawy and Werle [14] showed a different way of generating Allen-Southwell-Il'in scheme and derived a second-order, stable finite difference scheme for convection-diffusion problems and later Berger *et al.* [15] proved that the scheme is second-order accurate. Hegarty et al.[16] also proved the above result using the general convergence principle given by Miller. Roos in [17] described ten different approaches to generate uniformly convergent discretization schemes for one parameter singular perturbation problem. In the fitted mesh methods, various adaptive mesh constructions are given, for example, Bakhvalov [18] was the first who used the special grid for the solution of singularly perturbed differential equations. Later various adaptive meshes are given by various researchers, few are Vulanovic [19], Gartland [20], Shishkin [21, 22]. In the meshes given by Bakhvalov, Gartland, and Vulanovic the mesh size is uniform outside the layer(s) and in the decreasing/increasing form at the layer(s). However, Shishkin meshes are piecewise-uniform and one can also achieve uniform convergence on it.

In 1967, O'Malley [23] gave the asymptotic solution to two parameter singularly perturbed, problemsb(TPSPPs). Two decades later, some mathematicians presented the numerical techniques for the two parameter singularly perturbed boundary value problems. Relja Vulanovic [24] published a research article on numerical solution of singularly perturbed boundary value problem with two small parameters ε and μ with $\mu \le \varepsilon$. Two finite-difference schemes (up-wind and Samarskii's) are considered on a special non-uniform mesh. Up-wind scheme yields linear order of convergence and Samarskii's quadratic scheme is obtained. Roos and Uzelac [25] provide a priori bound of the solution and its derivatives for continuous problem. They used the streamline-diffusion finite element method on a Shishkin mesh and proved that the method is convergent independently of the perturbation parameters. In the literature so far convection-diffusion problems $(\mu = 1)$ and reaction-diffusion $(\varepsilon = 0)$ have been handled separately. In this paper, they provide a unified treatment of a two parameter problem for all possible classes of sub problems. Torsten [26] study a model linear convection-diffusion-reaction problem where both the diffusion term and the convection term are multiplied by small parameters ε and μ, respectively. Depending on the size of the parameters the solution of the problem may exhibit exponential layers at both end points of the domain. Sharp bounds for the derivatives of the solution are derived using a barrier function technique. These bounds are applied in the analysis of a simple upwind-difference scheme on Shishkin meshes. This method is established to be almost first-order convergent, independently of the parameters ε and μ. In the present study, we solved the singularly perturbed two parameter boundary value problems by using exponential splines. The boundary value problem is solved on a Shishkin mesh. In fact, the boundary value problem can be solved on any suitable adaptive mesh. The convergence analysis of the proposed method is studied. Numerical results are tabulated for different values of the perturbation parameters. From the numerical results, it is found that the method approximates the exact solution very well.

The paper is organized in the following manner: We stated the problem under consideration and construction of the numerical method is discussed in Section 2. To demonstrate the efficiency and applicability of the proposed methods, numerical experiments are carried out for four test problems and results are given in Section 3. The paper ends with conclusions.

2. Problem statement

We consider the two-parameter singularly perturbed convection-diffusion-reaction boundary value problem

$$Ly(x) = -\varepsilon y''(x) + \mu p(x) y'((x) + q(x) y(x) = r(x), \qquad x \in \Omega = (0,1),$$
$$y(0) = \alpha, \qquad y(1) = \beta, \tag{2.1}$$

with two small parameters $0<\varepsilon<<1$ and $0<\mu<<1$, such that $\frac{\mu^2}{\varepsilon}\to 0$ as $\varepsilon\to 0$. The functions $p(x)$, $q(x)$ and $r(x)$ are assumed to be sufficiently smooth with $p(x)\le -p^*<0$ and $q(x)\ge q^*>0$ for all $x\in[0,\ 1]$. This problem encompasses both the reaction-diffusion problem when $\mu = 0$ and the convection-diffusion problem when $\mu = 1$. It is well-known that standard numerical methods are unsuitable for singularly perturbed problems and fail to give accurate solutions.
We solve the above boundary value problem by using an exponential spline method on a Shishkin mesh.

2.1. Exponential spline method

Consider a uniform mesh Δ with nodal points x_i on the interval $[0,1]$ such that

$$\Delta: 0=x_1<x_2<....<x_{n-1}<x_n=1$$

where $x_i=ih$ and $(h=1/n)$; $i=1,2,.....n.$
To develop the method, we followed the steps given in [27].

Let $y(x)$ be the exact solution of the problem given by Eq.(2.1) and S_i be an approximation solution to $y_i(x)=y(x_i)$, obtained by the segment $Q_i(x)$ passing through the points (x_i,S_i) and (x_{i+1},S_{i+1}). Each mixed spline segment $Q_i(x)$ has the following form

$$Q_i(x)=a_ie^{k(x-x_i)}+b_ie^{-k(x-x_i)}+c_i(x-x_i)+d_i,\qquad i=0,1,2,3......\ n \tag{2.2}$$

where $a_i,\ b_i,\ c_i,\ d_i$ are constants and k is free parameter.
To obtain the necessary conditions for the coefficients introduced in Eq.(2.2) the segment value of $Q_i(x_i)$, $Q_i(x_{i+1}), Q_i^{(1)}, Q_{i+1}^{(1)}$ should be considered at the common node. Expressions for the four coefficients can be developed in terms of $S_i, S_{i+1}, M_i, M_{i+1}$ by defining

$$Q_i(x_i)=S_i,\qquad Q_i(x_{i+1})=S_{i+1},\qquad Q_i^{(2)}=M_i,\qquad Q_{i+1}^{(2)}=M_{i+1}. \tag{2.3}$$

From Eq.(2.2), we get

$$Q_i(x_i)=S_i=a_ie^{k(x-x_i)}+b_ie^{-k(x-x_i)}+c_i(x-x_i)+d,$$

$$Q_i(x_i)=S_i=a_i+b_i+d_i.$$

Now we calculate $Q_i(x_{i+1})$ from Eq.(2.2)

$$Q_i(x_{i+1})=S_{i+1}=a_ie^{k(x_{i+1}-x_i)}+b_ie^{-k(x_{i+1}-x_i)}+c_i(x_{i+1}-x_i)+d_i,$$

$$S_{i+1}=a_ie^{kh}+b_ie^{-kh}+c_ih+d_i.$$

We consider $k=\theta/h$, so $\theta=kh$, then

$$S_{i+1} = a_i e^{\theta} + b_i e^{-\theta} + c_i h + d_i . \tag{2.4}$$

Now, by taking differentiation of Eq.(2.2) with respect to x, we get

$$Q_i^{(1)}(x) = a_i k e^{k(x-x_i)} - b_i k e^{-k(x-x_i)} + c_i . \tag{2.5}$$

Now, by taking differentiation of Eq.(2.5) with respect to x, we get

$$Q_i^{(2)}(x) = a_i k^2 e^{k(x-x_i)} + b_i k^2 e^{-k(x-x_i)} .$$

We take $Q_i^{(2)}(x_i) = M_i$, so

$$Q_i^{(2)}(x_i) = M_i = a_i k^2 e^{k(x_i-x_i)} + b_i k^2 e^{-k(x_i-x_i)} ,$$

$$Q_i^{(2)}(x_i) = M_i = k^2 (a_i + b_i) . \tag{2.6}$$

Again, we know that

$$Q_i^{(2)}(x_{i+1}) = M_{i+1} ,$$

$$Q_i^{(2)}(x_{i+1}) = M_{i+1} = a_i k^2 e^{kh} + b_i k^2 e^{-kh},$$

$$Q_i^{(2)}(x_{i+1}) = M_{i+1} = k^2 \left(a_i e^{kh} + b_i e^{-kh} \right). \tag{2.7}$$

From Eq.(2.6)

$$k^2 (a_i + b_i) = M_i .$$

Therefore

$$a_i = \frac{M_i}{k^2} - b_i . \tag{2.8}$$

From Eq.(2.7), we have

$$M_{i+1} = k^2 \left(a_i e^{kh} + b_i e^{-kh} \right),$$

$$\frac{M_{i+1}}{k^2} = a_i e^{kh} + b_i e^{-kh} . \tag{2.9}$$

Now, we will put the value of a_i, from Eq.(2.8)

$$\frac{M_{i+1}}{k^2} = \left(\frac{M_i}{k^2} - b_i\right)e^{kh} + b_i e^{-kh},$$

$$\frac{M_{i+1}}{k^2} = \left(\frac{M_i}{k^2}\right)e^{kh} - 2b_i\left(\frac{e^{kh} - e^{-kh}}{2}\right),$$

$$\frac{M_{i+1}}{k^2} = \left(\frac{M_i}{k^2}\right)e^{kh} - 2b_i \sinh(kh),$$

$$b_i = \left(\frac{1}{k^2}\right)\left(\frac{M_i e^{kh} - M_{i+1}}{2\sinh(kh)}\right),$$

now, we consider $k = \theta h$, so $\theta = kh$

$$b_i = \left(\frac{h^2}{\theta^2}\right)\left(\frac{M_i e^{\theta} - M_{i+1}}{2\sinh(\theta)}\right) \tag{2.10}$$

now, by Eq. (2.8), we know that

$$a_i = \frac{M_i}{k^2} - b_i .$$

We put the value of b_i from Eq.(2.10), and $k = \theta / h$:

$$a_i = M_i\left(\frac{h^2}{\theta^2}\right) - \left(\frac{h^2}{\theta^2}\right)\left(\frac{M_i e^{i\theta} - M_{i+1}}{2\sinh(\theta)}\right),$$

$$a_i = \left(\frac{h^2}{\theta^2}\right)\left(\frac{M_i\, 2\sinh(\theta) - (M_i e^{i\theta} - M_{i+1})}{2\sinh(\theta)}\right),$$

$$a_i = \left(\frac{h^2}{\theta^2}\right)\left(\frac{M_{i+1} - M_i\, e^{-\theta}}{2\sinh(\theta)}\right). \tag{2.11}$$

Again, we have

$$d_i = S_i - a_i - b_i ,$$

now, by putting the value of a_i and b_i from Eq.(2.10) and (2.110), we get

$$d_i = S_i - \left(\frac{h^2}{\theta^2}\right)\left(\frac{M_{i+1} - M_i\, e^{-\theta}}{2\sinh(\theta)}\right) + \left(\frac{h^2}{\theta^2}\right)\left(\frac{M_i e^{\theta} - M_{i+1}}{2\sinh(\theta)}\right),$$

$$d_i = S_i - \left(\frac{h^2}{\theta^2}\right)\left(\frac{M_i(e^{\theta} - e^{-\theta}}{(e^{\theta} - e^{-\theta})}\right),$$

$$d_i = S_i - \left(\frac{h^2}{\theta^2}\right)M_i. \tag{2.12}$$

Again, we have from Eq.(2.4)

$$S_{i+1} = a_i e^{\theta} + b_i e^{-\theta} + c_i h + d_i,$$

$$c_i = \frac{1}{h}\left(S_{i+1} - a_i e^{\theta} - b_i e^{-\theta} - d_i\right).$$

now, by putting the value of a_i, b_i and d_i from Eqs (2.10)-(2.12), we get

$$c_i = \frac{1}{h}\left(S_{i+1} - S_i + \left(\frac{h^2}{\theta^2}\right)M_i - \left(\frac{h^2}{\theta^2}\right)\left(\frac{M_{i+1} - M_i\, e^{-\theta}}{2\sinh(\theta)}\right)e^{\theta} - \left(\frac{h^2}{\theta^2}\right)\left(\frac{M_i e^{\theta} - M_{i+1}}{2\sinh(\theta)}\right)e^{-\theta}\right),$$

$$c_i = \frac{1}{h}(S_{i+1} - S_i) + \left(\left(\frac{h^2}{\theta^2}\right)\right)(M_i - M_{i+1}).$$

Via straightforward calculation we obtained the value of a_i, b_i, c_i and d_i, as follows:

$$a_i = \left(\frac{h^2}{\theta^2}\right)\left(\frac{M_{i+1} - M_i\, e^{-\theta}}{2\sinh(\theta)}\right), \qquad b_i = \left(\frac{h^2}{\theta^2}\right)\left(\frac{M_i e^{\theta} - M_{i+1}}{2\sinh(\theta)}\right),$$

$$c_i = \frac{1}{h}(S_{i+1} - S_i) + \left(\left(\frac{h^2}{\theta^2}\right)\right)(M_i - M_{i+1}), \qquad d_i = S_i - \frac{h^2 M_i}{\theta^2}$$

where

$$\theta = kh \text{ and } i = 0, 1, 2, \ldots. n.$$

Using the continuity of the first derivative at the point (x_i, S_i) where $Q_i^{(1)}$ and $Q_{i+1}^{(1)}$ the following relation for $i = 0, 1, 2, \ldots.. n-1$ is obtained

$$\left(S_{i+1} - 2S_i + S_{i-1}\right) = h^2\left(\alpha M_{i+1} + \beta M_i + \gamma M_{i-1}\right) \tag{2.13}$$

where

$$\alpha=\left(\frac{\sinh(\theta)-\theta}{\theta^2\sinh(\theta)}\right), \qquad \beta=\left(\frac{\left(2\theta\cosh(\theta)-2\sinh(\theta)\right)}{\theta^2\sinh(\theta)}\right).$$

When $k\to 0$ and $\theta\to 0$ then $(\alpha,\beta)=\left(\frac{1}{6}\right)(1,4)$ and the relation defined by Eq.(2.13) reduces to the following ordinary cubic spline relation:

$$\left(S_{i+1}-2S_i+S_{i-1}\right)=\frac{h^2}{6}\left(M_{i+1}+4M_i+M_{i-1}\right), \tag{2.14}$$

at the point x_i the proposed singularly problem may be described by:

$$M_i=\frac{1}{\varepsilon}\left(\mu p_i S_i^{(1)}+r_i\right) \tag{2.15}$$

where

$$S_i^{(1)}=\frac{S_{i+1}-S_{i-1}}{2h}, \qquad S_{i+1}^{(1)}=\frac{3S_{i+1}-4S_i+S_{i-1}}{2h}, \qquad S_{i-1}^{(1)}=\frac{-S_{i+1}+4S_i-3S_{i-1}}{2h}.$$

Substituting Eq.(2.15) into Eq.(2.13), we get the following

$$\left(S_{i+1}-2S_i+S_{i-1}\right)=\frac{h^2}{\varepsilon}\left[\alpha\left(\mu p_{i+1}S_{i+1}^{(1)}+r_{i+1}\right)+\beta\left(\mu p_i S_i^{(1)}+r_i\right)+\alpha\left(\mu p_{i-1}S_{i-1}^{(1)}+r_{i-1}\right)\right].$$

Now we will put the values $S_{i+1}^{(1)}$, $S_i^{(1)}$, $S_{i-1}^{(1)}$

$$\varepsilon\left(S_{i+1}-2S_i+S_{i-1}\right)=\mu h^2\left[\alpha\, p_{i+1}\left(\frac{3S_{i+1}-4S_i+S_{i-1}}{2h}\right)+\right.$$
$$\left.+\beta p_i\left(\frac{S_{i+1}-S_{i-1}}{2h}\right)+\alpha p_{i-1}\left(\frac{-S_{i+1}+4S_i+S_{i-1}}{2h}\right)+h^2\left(\alpha\, r_{i1}+\beta r_i+\alpha r_{i-1}\right)\right].$$

Simplifying, we get

$$-\varepsilon\left(S_{i-1}-2S_i+S_{i+1}\right)+\frac{\mu h}{2}\left[D_i S_{i-1}+E_i S_i+A_i S_{i+1}\right]=-h^2\left(\alpha r_{i-1}-\beta r_i+\alpha r_{i+1}\right) \tag{2.16}$$

where $i = 1,2,3,.....,n-1$ and

$$A_i=-\alpha p_{i-1}+\beta p_i+3\alpha p_{i+1},$$

$$D_i=-3\alpha p_{i-1}-\beta p_i+\alpha p_{i+1},$$

$$E_i=4\alpha\left(-p_{i+1}+p_{i-1}\right).$$

The above system gives $(n-1)$ linear algebraic equations with $(n+1)$ unknowns S_i. We use two given boundary conditions, so that we have $(n+1)$ equations with $(n+1)$ unknowns. To obtain the numerical solution, we solved this tridiagonal system by using Thomas Algorithm.

2.2. Mesh selection strategy

We know that an equidistant mesh cannot attain convergence at all mesh points uniformly in ε and μ. If its coefficients have an exponential property, then the scheme can attain convergence at all mesh points uniformly in ε and μ. Therefore, unless a specially chosen mesh is used, we cannot obtain a parameter uniform convergence at all the mesh points. So a simple non-uniform mesh, namely a piecewise uniform mesh discussed in [28] is enough for the construction of a parameter uniform method. It is fine near layers but coarser otherwise. We cannot say that these piecewise uniform meshes are optimal in any sense. It is useful because it is simple and adequate for solving a wide variety of singularly perturbed problems. To use the Shishkin mesh one should have a priori knowledge about the location and nature of the layers. To obtain the discrete counterpart of the two-parameter singularly perturbed boundary value problem (1), firstly the considered mesh discretized the domain $\Omega=[0,\ 1]$ into three subintervals

$$A_0=[0,\gamma_1],\quad A_c=[\gamma_1,1-\gamma_2]\quad \text{and}\quad A_1=[1-\gamma_2,1]$$

where transition parameters are given by

$$\gamma_1=\min\left(\frac{1}{4},\frac{2}{\psi_1}\ln n\right),\qquad \gamma_2=\min\left(\frac{1}{4},\frac{2}{\psi_2}\ln n\right),$$

with n to be the number of subdivision points of the interval $[0,1]$ and we place $(n/4)$, $(n/2)$, $(n/4)$ mesh points, respectively, in $[0,\gamma_1],[\gamma_1,1-\gamma_2],\ [1-\gamma_2,1]$. Accordingly, the resulting piecewise uniform Shishkin mesh may be represented by:

$$h=\begin{cases} h_1=\dfrac{4\gamma_1}{n} & x_i=x_{i-1}+h_1\ \ for\ \ i=1,2,3,.....n/4,\\[2ex] h_2=\dfrac{2(1-\gamma_1-\gamma_2)}{n} & x_i=x_{i-1}+h_2\ \ for\ \ i=1,2,3,.....3n/4,\\[2ex] h_3=\dfrac{4\gamma_2}{n} & x_i=x_{i-1}+h_3\ \ for\ \ i=1,2,3,.....n. \end{cases}$$

3. Numerical results

To check the applicability of the proposed method, we applied it to three examples. The maximum absolute errors and corresponding rate of convergence has been tabulated. For the examples, where the numerical solutions are not available, we used the double mesh principle to calculate the maximum absolute errors.

Example 1 [29]. Consider the singularly perturbed boundary value problem

$$-\varepsilon y'' + \mu y' + y = \cos(\pi x), \qquad x \in (0,1), \qquad y(0) = 0, \qquad y(1) = 0.$$

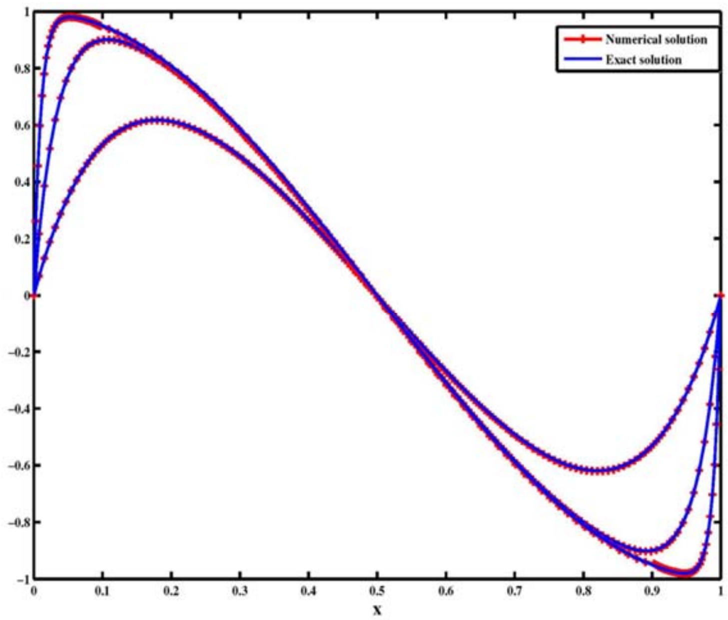

Fig.1. Exact and approximate solution for Example *1* at various values of ε and $\mu = 10^{-6}$.

Table 1. Maximum absolute error and rate of convergence (ROC) for $\mu = 10^{-6}$ and for various values of ε.

$\mu = 10^{-6}$				
ε	N=64	N=128	N=256	N=512
10^{-2} ROC	7.5210×10^{-2} 1.0194	3.7161×10^{-3} 0.9615	1.9023×10^{-3} 3.2765	1.7136×10^{-4}
10^{-3} ROC	5.1211×10^{-2} 0.9823	4.1345×10^{-3} 1.0356	2.1035×10^{-3} 3.3456	1.9245×10^{-4}
10^{-4} ROC	1.3235×10^{-2} 0.9789	6.9485×10^{-3} 0.8606	6.3368×10^{-3} 3.2963	5.3214×10^{-4}
10^{-5} ROC	1.5574×10^{-2} 1.0093	4.6923×10^{-3} 1.3120	3.8255×10^{-3} 3.2643	3.4569×10^{-4}
10^{-6} ROC	1.6141×10^{-2} 1.0090	6.6136×10^{-3} 1.0020	4.2912×10^{-3} 3.1341	3.2145×10^{-4}

The exact solution is given by

$$\rho_1 \cos(\pi x) + \rho_2 \sin(\pi x) + \psi_1 e^{\lambda_1 x} + \psi_2 e^{-\lambda_1 (1-x)}.$$

where

$$\rho_1 = \frac{(\varepsilon\pi^2 + 1)}{(\mu^2\pi^2 + (\varepsilon\pi^2 + 1)^2)}, \quad \rho_2 = \frac{\mu}{(\mu^2\pi^2 + (\varepsilon\pi^2 + 1)^2)}, \quad \psi_1 = \frac{-\rho_1(1 + e^{-\lambda_2})}{(1 - e^{\lambda_1 - \lambda_2})},$$

$$\psi_2 = \frac{-\rho_1(1 + e^{-\lambda_1})}{(1 - e^{\lambda_1 - \lambda_2})}, \quad \lambda_1 = \frac{\mu - \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon}, \quad \lambda_2 = \frac{\mu + \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon}.$$

Maximum absolute errors and the corresponding rate of convergence (ROC) for this example are given in Tab.1. The exact solution and approximate solution for different values of ε and $\mu = 10^{-6}$ are plotted in Fig.1.

Example 2. Consider the singularly perturbed boundary value problem

$$-\varepsilon y'' + y' + y = 1, \quad x \in (0,1), \quad y(0) = 0, \quad y(1) = 0.$$

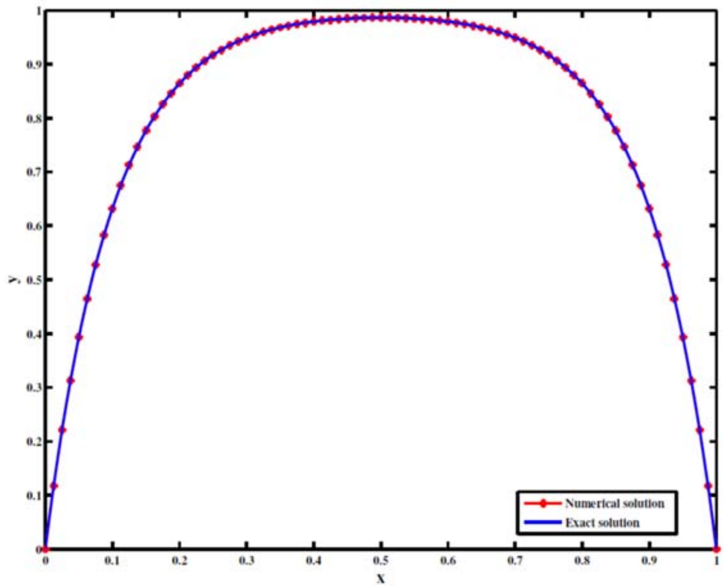

Fig.2. Exact and approximate solution for Example 2 when $\varepsilon = 10^{-7}$ and $\mu = 10^{-2}$.

The exact solution is given by

$$y(x) = \frac{\left(e^{\lambda_2} - 1\right)}{\left(e^{\lambda_1} - e^{\lambda_2}\right)} + \frac{\left(1 - e^{\lambda_1}\right)e^{\lambda_2 x}}{\left(e^{\lambda_1} - e^{\lambda_2}\right)} + 1$$

where

$$\lambda_1 = \frac{1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon}, \quad \lambda_2 = \frac{1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon}.$$

Maximum absolute errors and the corresponding rate of convergence (ROC) for Example 2 are given in Tab.2. The exact solution and approximate solution for different values of ε and $\mu = 10^{-2}$ are plotted in Fig.2.

Table 2. Maximum absolute error and rate of convergence for $\mu = 10^{-2}$ and for various values of ε.

$\mu = 10^{-2}$				
ε	*N=40*	*N=80*	*N=160*	*N=360*
10^{-3} ROC	1.3612×10^{-2} 1.10401	6.6194×10^{-3} 2.3222	1.3236×10^{-3} 2.0190	3.2657×10^{-4}
10^{-4} ROC	1.3276×10^{-2} 0.9789	4.6051×10^{-3} 0.8606	9.3665×10^{-4} 3.2963	2.3146×10^{-4}
10^{-5} ROC	1.2657×10^{-4} 1.0368	6.1690×10^{-5} 1.1109	3.0612×10^{-5} 2.6920	1.1843×10^{-6}
10^{-6} ROC	1.2191×10^{-5} 1.8852	3.3002×10^{-6} 2.2609	6.8858×10^{-7} 2.1952	1.5036×10^{-7}
10^{-7} ROC	6.8819×10^{-6} 1.6222	2.2355×10^{-6} 2.3603	4.3537×10^{-7} 2.8023	6.2415×10^{-8}

Example 3. Consider the singularly perturbed boundary value problem

$$-\varepsilon\, u''(x)+\mu\left(3-2x^2\right)u'(x)+u(x)=(1+x)^2, \qquad x\in(0,1), \qquad u(0)=0, \qquad u(1)=0.$$

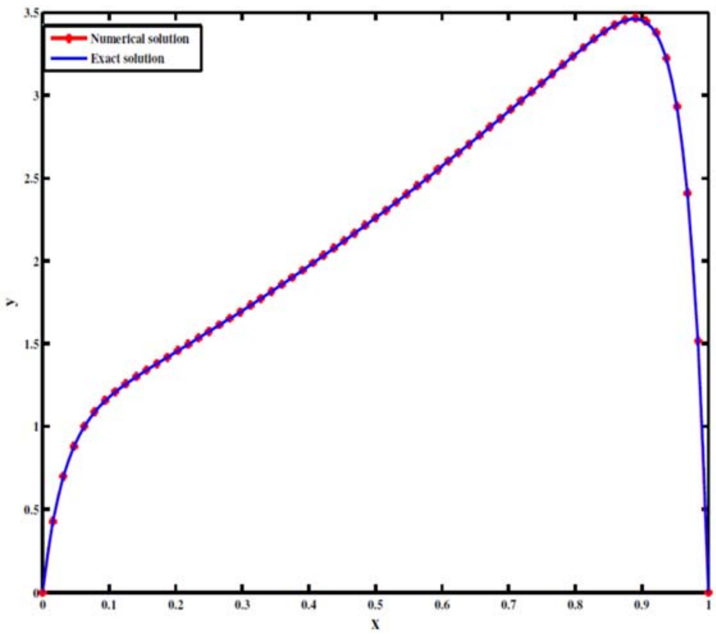

Fig.3. Numerical solution of Example 3 when $\varepsilon = 10^{-9}$ and $\mu = 10^{-3}$.

Since the exact solution is not available for this example, we used the double mesh principle to calculate numerical solution.

Table 3. Maximum absolute error and rate of convergence for $\mu = 10^{-3}$ and for various values of ε .

$\mu = 10^{-3}$				
ε	*N=64*	*N=128*	*N=256*	*N=512*
10^{-5} ROC	1.3124×10^{-4} 1.0368	6.1690×10^{-5} 1.1109	3.0612×10^{-5} 2.6920	1.1843×10^{-6}
10^{-6} ROC	1.2191×10^{-5} 1.8852	3.3002×10^{-6} 2.2609	6.8858×10^{-7} 2.1952	1.5036×10^{-7}
10^{-7} ROC	6.8819×10^{-6} 1.6222	2.2355×10^{-6} 2.3603	4.3537×10^{-7} 2.8023	6.2415×10^{-8}
10^{-8} ROC	1.4165×10^{-2} 1.6858	4.3655×10^{-3} 1.5236	1.5465×10^{-3} 1.5149	6.5989×10^{-4}
10^{-9} ROC	1.3896×10^{-3} 1.6859	3.8645×10^{-3} 1.5236	1.2364×10^{-3} 3.2537	6.7099×10^{-4}

4. Conclusion

In this paper, a numerical method is studied to solve two-parameter singularly perturbed linear boundary value problems. This method is based on exponential spline with piecewise uniform Shishkin mesh. The method is convergent for all perturbation parameters ε and μ. From the table, it can be observed that, as the step sizes are decreasing, the maximum absolute errors are decreasing, which shows the numerical convergence of the proposed method. It has been found that the proposed algorithm gives highly accurate numerical results and higher order of convergence.

Nomenclature

$a_i,\ b_i,\ c_i,\ d_i$ – constants and

k – free parameter.

ε , μ – small positive parameter

α,β – real constants

γ_1,γ_2 – transition parameters

References

[1] Hirsh C. (1988): *Numerical Computation of Internal and External Flows*.– John Wiley & Sons, Chischester, vol.1.

[2] Hirsh C. (1990): *Numerical Computation of Internal and External Flows*.– John Wiley & Sons, Chischester, vol.2.

[3] Ewing R.E. (1983) : *The Mathematics of Reservoir Simulation*.– SIAM, Philadelphia.

[4] Jacob M. (1959): *Heat Transfer*.– John Wiley & Sons, New York.

[5] Baumert H., Braun P., Glos E., Muller W.D. and Stoyans G. (1981): *Modelling and computation of water quality problems in river networks*.– Lecture Notes in Control and Information Sciences, Springer-Velag, Berlin, pp.482-491.

[6] Markowich P.A., Ringhofer C.A., Selberherr S. and Lentini M. (1983): *A singular perturbation approach for the analysis of the fundamental semiconductor equations*.– IEEE Trans. Electron Device, vol.30, pp.1165-1180.

[7] Van Roosbroeck W.V. (1950): *Theory of flows of electrons and holes in germanium and other semiconductors*.– Bell Syst. Tech. J., vol 29, pp.560-607.

[8] Murray J.D. (1977): *Lectures on Nonlinear Differential Equation Models in Biology*.– Clarendon Press, Oxford.

[9] Weekman V.W. and Gorring R.L. (1965): *Influence of volume change on gas-phase reactions in porous catalysts*.– J. Catalysis, vol.4, pp.260-270.

[10] Gilberg D. and Trudinger N. (1977): *Elliptic Partial Differential Equations of Second Order*.– *Springer-Verlag*, p.518.

[11] Allen D. N. and de G.Southwell R.V. (1955): *Relaxation methods applied to determine the motion in two dimensions of a viscous fluid past a fixed cylinder*.– Quart. J.Mech. Appl. Math., vol.8, pp.129-145.

[12] Il'in A.M. (1969): *A difference scheme for a differential equation with a small parameter multiplying the highest derivative (in Russian)*.– Mat. Zametki, vol.6, pp. 237-248, English translation: Math. Notes., vol.6, No.2, pp.596-602.

[13] Kellogg R. B. and Tsan A.(1978): *Analysis of some difference approximations for a singular perturbation problem without turning points*.– Math. Comp., vol.32, No.144, pp.1025-1039.

[14] El-Mistikawy T.M. and Werle M.J. (1978): *Numerical method for boundary layers with blowing – the exponential box scheme*.– AIAA J., vol.16, pp.749-751.

[15] Berger A.E., Soloman J.M. and Ciment M. (1981): *An analysis of a uniformly accurate difference method for a singular perturbation problem*.– Math. Comp., vol.37, No.155, pp.79-94.

[16] Hegarty A.F., Miller J.J.H. and O'Riordan E. (1980): *Uniform second order difference schemes for singular perturbation problems. Boundary and interior layers computational and asymptotic method*s.– Proc. Conf., Trinity College, Dublin, pp.301-305.

[17] Roos H.G. (1994): *Ten ways to generate the Il'in and related schemes*.– J. Comput. Appl. Math., vol.53, No.1, pp.4359.

[18] Bakhvalov N.S. (1969): *On the optimization of the methods for solving boundary value problems in the presence of a boundary layer (in Russian)*.– Z. Vycisl. Mat i Mat. Fiz., vol.9, pp.841-859.

[19] Vulanovic R., Herceg D. and Petrovic N. (1986): *On the extrapolation for a singularly perturbed boundary value problem*.– Computing, vol.36, pp.69-79.

[20] Gartland E.C. (1988*)*: *Graded-mesh difference schemes for singularly perturbed two-point boundary value problems*.– Math. Comput., vol.51, No.184, pp.631-657.

[21] Shishkin G.I. (1989): *Approximation of solutions of singularly perturbed boundary value problems with a parabolic boundary layer*.– U.S.S.R. Comput. Math. and Math. Phys., vol.29, No.4, pp.1-10.

[22] Miller J.J.H., O'Riordan E. and Shishkin G.I. (1996): *Fitted Numerical Methods for Singular Perturbation Problems. Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions*.– World Scientific, Singapore, p.192.

[23] O'Malley Jr. R.E. (1967): *Singular perturbations of boundary value problems for linear ordinary differential equations involving two-parameters*.– J. Math. Anal. Appl., vol.19, pp.291-308.

[24] Vulanovic R. and Ohio C. (2001): *A higher order scheme for quasilinear boundary value problems with two small parameters*.– Computing, vol.67, pp.287-303.

[25] Roos H.G. and Uzela Z. (2003): *The SDFEM for a convection-diffusion problem with two small parameter*.– Comput. Methods Appl. Math., vol.3, pp.443-458.

[26] Linss T. (2010): *A posterior error estimation for a singularly perturbed problem with two small parameters*.– Int. J. Numer. Anal. Model., vol.7, pp.491-506.

[27] Zahra W.K. and Ashraf M. EI Mhlawy (2013): *Numerical solution of two-parameter singularly perturbed boundary value problems via exponential spline*.– King Saud University, vol.25, pp.201-208.

[28] Linss T. and Roos H-G. (2004): *Analysis of a finite-difference scheme for a singularly perturbed problem with two small parameters*.– J.Math.Anal.Apppl., vol.289, pp.424-442.

[29] Kadalbajoo M.K. and Yadaw A.S. (2008): *B-spline collection method for a two-parameter singularly perturbed convection-diffusion boundary value problem*.– Appl. Math. Comput., vol.201, pp.504-513.

Received: January 2, 2021

Revised: March 3, 2021