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HEAT TRANSFER MODELLING IN AN ANNULAR DISC UNDER HEATING AND COOLING PROCESSES WITH STRESS ANALYSIS

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The goal of this effort is to determine the interaction among the heating and cooling processes in order to understand how solids behave when subjected to temperature changes. In this instance, the temperature, displacement, and stress relations are determined analytically and numerically while a thin annular disc is subjected to both the heating and cooling processes. The ability of a material to withstand stress is essential for the design of diverse mechanical structures that aim to enhance performance, durability, characteristics, and strength. This ability is demonstrated in many physical processes where the material structure crosses over into heating and cooling processes. Furthermore, memory derivatives used in the modelling of heat transfer equations more accurately depict the memory behaviour of an imagined disc and explain its physical significance.

Key words: memory related derivatives, heating and cooling process, integral transform, temperature, stress functions.

1. Introduction

Different solids' thermoelasticity issues during heating and cooling operations have a direct impact on their structural characteristics. The significance of integrability, stress-bearing capability, and performance stems from the fact that both processes include particle position changes. It should be emphasized that a large number of scholars have only concentrated on understanding the effects of an additional partially dispersed heat source on deformation, temperature change, and stress. However, very little research has been done that takes into account the effects of processing heat and cold in a solid body. A small number of published research works by a well-known author are listed below.

Khobragade and Deshmukh [1] provided an analytical solution to a thermoelastic problem involving a partially dispersed heat supply for a thin plate. By using integral transformations, Gaikwad [2] ascertained the disc problem's thermal behaviour caused by an axisymmetric, partly dispersed heat source. Ishihara *et al.* [3] used the analytical method to theoretically assess the thermo-elastoplastic deformation of the bending plate. Ootao *et al.* [4] measured the stresses and temperature change in a two-dimensional laminated beam with a heat source that is

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partially dispersed. Because of a partial heat supply, Tanigawa and Yoshinobu [5] effectively examined the theoretical effects of stress in a three-dimensional functionally graded problem for a rectangular plate.

These days, MDD is a novel area in the calculus of fractions that is still growing in popularity in recent years. Despite being specified on an interval, the fractional derivative mostly captures the change that occurs locally. It is considerably easier to understand the physical meaning of MDD than it is for the fractional derivative. The dependent weight is reflected by the kernel function, and the time-delay indicates how long the memory effect lasts. Existing research indicates that the MDD is better suited for temporal modelling. Memory dependent derivative have uses throughout particle physics, vibration mechanics, thermoelasticity, and thermoelectricity.

Wang and Li [6] created the novel idea of memory response by proposing a memory-dependent derivative, which has a physically stronger meaning and can be used in practical applications, instead of the Caputo-type derivative of fractional order. Yu, *et al.* [7] examined how the memory idea is used to describe the practical response and superiority of materials over fractional ones in the theory of generalised thermoelasticity. Sur and Kanoria [8] investigated a new mathematical model for magneto-thermoelasticity and commented about the transient occurrences in generalised thermoelasticity for a thick plate reinforced with fibres and heated by a heat source. Al-Jamel *et al.* [9] defined linked displacement and discussed damping in an oscillatory system in relation to a memory-dependent derivative. In order to create the generalized time delay higher-order model, Abouelregal *et al.* [10] took memory derivatives into account and discussed the resulting findings with heat sources.

El-Karamany *et al.* [11] developed the constitutive relationship for thermoelastic diffusion in the framework of a novel generalised thermoelasticity theory that was influenced by kernel functions and time delay factors. By taking into account the mixed initial-boundary value problem for isotropic media, Sarkar and Mukhopadhyay *et al.* [12] provided the theoretical contribution theorem for generalised thermoelasticity with memory-dependent derivative. Li and He [13] performed an analytical solution of the coupled controlling equations, which were then computed numerically and subjected to freely selected time-delay factors and kernel functions. Using a new theory of generalised thermoelasticity, Sarkar *et al.* [14] examined the propagation consequence of a magneto-thermoelastic relationship to memory-dependent derivative in an isotropic homogeneous completely conducting multifaceted semi-infinite environment. Lamba [15] explored the memory-dependent thermoelastic response by discussing the internal heat source effect on cylindrical bodies under radiation-like limits. Verma *et al.* [16] successfully addressed a variety of thermal behaviours when preparing the model of the hygrothermal problem with fraction order theory under linked and uncoupled thermoelasticity. Yadav *et al.* [17] investigated the significant memory dependent triple-phase-lag thermo-elasticity and discussed the reflection of plane waves in a thermo-diffusion medium. Lamba [18] recently developed thermoelastic model of fractional order by considering functionally graded materials.
Thakare *et al.* [19] investigated the response of a fractional order thick cylinder with non-homogeneous material properties under heat generation and analysed the resultant stress functions with fractional order variations. Lamba and Khobragade [20] used the integral transformation technique to solve the thermal distribution problem in a rectangular region and determine the stress impact with known applied constraints on boundaries. Kumar and Kamdi [21] investigated the problem of a finite-length cylinder under the framework of fractional thermoelasticity with convective heating. Further Lamba *et al.* [22] focus on the brief study to know the response of a layer under fractional parameters.

Under memory-dependent derivative thermoelasticity, Yadav *et al.* [23] examined the reflection amplitude in a nonlocal porous-thermo-micropolar diffusion half-space. Yadav and Schnack [24] studied the phenomenon of reflection in a magnetized, conductive thermo-triclinic solid half-space using the memory-dependent nonlocal magneto-thermoelasticity theory. Yadav [25-28] conducted a successful study on plane wave reflection in a fractional-order thermoelasticity theory. The new viewpoint on the conventional resolutions of the nonlinear time-fractional partial differential equation problem was effectively established by Ahmad *et al.* [29-31]. Some same work related to the field of investigation is cited in the references [32-39]. Very recently, Sheikh *et al.* [40] investigated an innovative hygrothermal model with MDD and higher-order temporal derivatives with three-phase delays using the theory of integral transformations.

In order to ascertain the memory behaviour of a thin annular disc subjected to heating and cooling, the authors of this paper employed the integral transformation technique. The disc undergoing the heating process is subjected to partial heating at the inner curved surface $-Q_0/\lambda\, g(z,t)$ and the function $f(z,t)$ is prescribed at the outer curved surface, while the cooling process removes the previously mentioned partial heating function from the inner curved surface and subjects the outer curved surface to zero. The heat transfer equation assumes a memory derivative in order to determine the precise retarded response that occurs throughout the physical processing of materials. Furthermore, by taking into account the material properties of aluminium metal, the mathematical representation of temperature, displacement, and stresses is determined analytically and plotted in terms of appearance.

2. Construction of the mathematical problem

Assume that the radius changes from $r=a$ to $r=b$ and that the annular disc has a thickness of $2h$. The disc's starting point temperature is equal to that of the surrounding medium, which is maintained at that same level from $t=0$ to $t=t_0$. A partially dispersed, axisymmetric source of heat $Q_0\, g(z,t)/\lambda$ is applied to the disc from the exterior. The heat source is then turned removed, and then the surrounding medium cools the disc. In Fig.1, the geometrical aspect is depicted.

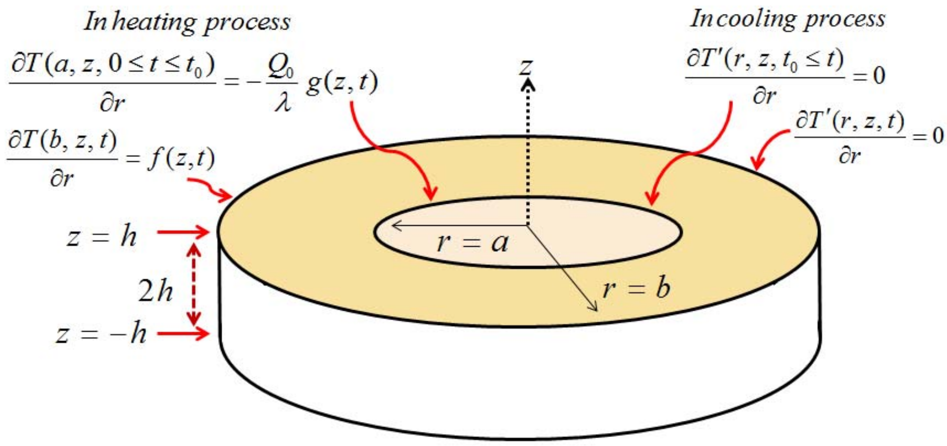

Fig.1. Concern design and the impact of MDD responsiveness during heating and cooling.

2.1. Formulation of heat equation with impact of MDD

For a homogeneous and isotropic solid, the link involving the flow of heat and the temperature gradient in the basic theory of heat transfer is [39]:

$$q=-k\nabla T \tag{2.1}$$

where q denote heat flux vector, k is thermal conductivity, ∇ is the spatial gradient operator, and T is absolute temperature.

To provide a better understanding of how heat spreads through materials, numerous generalised models of thermal conduction have been constructed. Of the models, the Cattaneo and Vernotte (CV) framework [11] is described as:

$$q+\tau\frac{\partial q}{\partial t}=-k\nabla T \tag{2.2}$$

where τ is the thermal relaxation time of CV model.

The first-order MDD in the flow of heat was utilized by Yu *et al.* [7] to denote the novel memory-dependent CV theory as follows:

$$q+\tau D_{\Omega}q=-k\nabla T\ . \tag{2.3}$$

When there is no thermal origin, the energy conservation formula is expressed as:

$$-\nabla q=\rho_m c_E\frac{\partial T}{\partial t} \tag{2.4}$$

where ρ_m and c_E are, respectively, mass density and specific heat capacity.

The generalised thermal transfer formula, including MDD, for an axially symmetric scenario with isotropic and elastic material characteristics in the cylinder-shaped dimensions can be obtained by inserting Eq.(2.3) into Eq.(2.4).

$$k\nabla^2 T=(1+\tau D_{\Omega})\rho_m c_E\frac{\partial T}{\partial t} \tag{2.5}$$

where Laplace equation for finite length disc is $\nabla^2=\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{\partial^2}{\partial z^2}$.

For convenience, establish a number of non-dimensional parameters:

$$r'=\frac{r(t',\tau',\Omega')}{r_0}=\frac{1}{\rho_m c_E r_0^2}(t,\tau,\Omega),\quad z'=\frac{z(t',\tau',\Omega')}{z_0}=\frac{1}{\rho_m c_E r_0^2}(t,\tau,\Omega),\quad T'=\frac{T}{T_0}\ .$$

By utilizing the non-dimensional parameters previously mentioned, the equations that govern Eq.(2.5) can be recast (falling the primes for simplicity) [15].

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=(1+\tau D_{\Omega})\frac{1}{k}\frac{\partial T}{\partial t} \tag{2.6}$$

where the memory-dependent derivative of order m of $T(r,z,t)$ is:

$$D_{\Omega}^{m}T(r,z,t)=\frac{\partial^{m-1}}{\partial t^{m-1}}D_{\Omega}T(r,z,t)=\frac{1}{\Omega}\int_{t-\Omega}^{t}K(t-\xi)\frac{\partial^m T(r,z,\xi)}{\partial \xi^m}d\xi \tag{2.7}$$

here, the Kernel function $K(t-\xi)$ and time delay Ω are chosen arbitrarily in order to capture materials real behaviours.

In general, the memory effect requires weight $0\le K(t-\xi)\le 1$ for $\xi\in[t-\Omega,t]$ so that the magnitude of the memory-dependent derivative $D_{\Omega}T(r,z,t)$ is usually smaller than that of the common partial derivative $\dfrac{\partial T(r,z,t)}{\partial t}$.

2.2. Stress-displacement relationship

The differentiable form of the displacement-temperature equation is as outlined below [36]:

$$\frac{\partial^2 U(r,z,t)}{\partial r^2}+\frac{1}{r}\frac{\partial U(r,z,t)}{\partial r}=(1+\nu)\,a_t\,T(r,z,t) \tag{2.8}$$

with $$U(a,z,t)=0 \text{ and } U(b,z,t)=0, \tag{2.9}$$

here ν and a_t are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the disc respectively.

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given as [36]:

$$\sigma_{rr}=-2\mu\frac{1}{r}\frac{\partial U}{\partial r}, \tag{2.10}$$

$$\sigma_{\theta\theta}=-2\mu\frac{\partial^2 U}{\partial r^2} \tag{2.11}$$

where μ is the Lame's constant, white each of the stress functions σ_{rz}, σ_{zz} and $\sigma_{\theta z}$ are zero within the disc in the plane state of stress.

The mathematical formulas Eq.(2.6) through Eq.(2.11) provide an analytical description of the subject that is being investigated.

3. Boundary constraints in case of partial heating process

The initial temperature of the disc is the same as the temperature of the surrounding medium, which is kept constant for the time $t=0$ to $t=t_0$, the disc is subjected to a partially distributed and axisymmetric heat supply $Q_0\,g(z,t)/\lambda$ from the outer surface. In this case the heat Eq.(2.6) with MDD is subjected to the following initial and boundary conditions:

$$T(r,z,0)=0; \tag{3.1}$$

$$\frac{\partial T(a,z,t)}{\partial r}=-\frac{Q_0}{\lambda}\,g(z,t);\quad 0\le t\le t_0, \tag{3.2}$$

$$\frac{\partial T(b,z,t)}{\partial r}=f(z,t), \tag{3.3}$$

$$T(r,-h,t)+k_1\frac{\partial T(r,-h,t)}{\partial z}=0, \tag{3.4}$$

$$T(r,h,t)+k_2\frac{\partial T(r,h,t)}{\partial z}=0 \tag{3.5}$$

where k_1 and k_2 are the radiation constant on the two plane surfaces.

3.1. The solution of the MDD heat equation with a partial heating process

Utilizing Eqs (3.4-3.5) and the finite Marchi-Fasulo integral transform [37] on mathematical equations (2.6), (3.1), (3.2), and (3.3) yields:

$$\frac{\partial^2 \overline{T}}{\partial r^2}+\frac{1}{r}\frac{\partial \overline{T}}{\partial r}-a_n^2\,\overline{T}=(1+\tau D_\Omega)\frac{1}{k}\frac{\partial \overline{T}}{\partial t}. \tag{3.6}$$

Along the equation's mathematical solutions are represented by the Eigen values a_n as:

$$\begin{aligned}&[\alpha_1 a\cos(ah)+\beta_1\sin(ah)][\beta_2\cos(ah)+\alpha_2 a\sin(ah)]=\\&=[\alpha_2 a\cos(ah)-\beta_2\sin(ah)][\beta_1\cos(ah)-\alpha_1 a\sin(ah)].\end{aligned}$$

In the domain of the Marchi-Fasulo integral transform, restriction and initially constraints become

$$\overline{T}(r,n,0)=0; \tag{3.7}$$

$$\frac{\partial \overline{T}(a,n,t)}{\partial r}=-\frac{Q_0}{\lambda}\,\overline{g}(n,t);\quad 0\le t\le t_0, \tag{3.8}$$

$$\frac{\partial \overline{T}(b,n,t)}{\partial r}=\overline{f}(n,t) \tag{3.9}$$

where $\overline{T}$ denotes the Marchi-Fasulo transform of T and n is the Marchi-Fasulo transform parameter α_1,α_2, β_1 and β_2 are constants.

Subsequently, if one applies the integral transformation of Laplace [38] to Eq.(3.6) on each side, the result is Eq.(3.10).

$$\frac{\partial^2 \overline{T}^*(r,n,s)}{\partial r^2}+\frac{1}{r}\frac{\partial \overline{T}^*(r,n,s)}{\partial r}-q^2\overline{T}^*(r,n,s)=0 \tag{3.10}$$

where

$$G=\frac{\tau}{\Omega}\left\{\left(1-e^{-s\Omega}\right)\left(1-\frac{2l_2}{\Omega s}+\frac{2l_1^2}{\Omega^2 s^2}\right)-\left(l_1^2-2l_2+\frac{2l_1^2}{\Omega s}\right)e^{-s\Omega}\right\},\qquad q^2=a_n^2+\frac{(1+G)s}{k}.$$

where l_1 and l_2 are constants.

In the Laplace transforms domain, boundary circumstances evolve into following form:

$$\frac{\partial \overline{T}^*(a,n,s)}{\partial r}=-\frac{Q_0}{\lambda}\,\overline{g}^*(n,s), \tag{3.11}$$

$$\frac{\partial \overline{T}^*(b,n,s)}{\partial r}=\overline{f}^*(n,s) \tag{3.12}$$

where $\overline{T}^*$ denotes the Laplace transform of $\overline{T}$ and s is a Laplace transform parameter.

After figuring out the solution to Eq.(3.10), which is expressed in terms of Bessel's function with unknown constants whose values are easily determinable by using the boundary constraints specified in Eq.(3.11) and

Eq.(3.12). And lastly, the equation of temperature distribution in the Laplace domain can be obtained by changing these constant values and inverting for March-Fasulo transform.

$$
\begin{aligned}
T^{*}(r,z,s) &= \frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\frac{I_0'(qb)k_0(qr)-k_0'(qb)I_0(qr)}{q\,[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\,\overline{g}^{*}(n,s)+\\
&+\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\frac{I_0'(qa)k_0(qr)-k_0'(qa)I_0(qr)}{q\,[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\,\overline{f}^{*}(n,s).
\end{aligned}
\tag{3.13}
$$

The response to heat formula Eq.(2.6) with the influence of memory-dependent implications in the transform's Laplace domain is given by calculation Eq.(3.13) above.

4. Boundary constraints in case of cooling process

In this instance, the disc is cooled by its surrounding media once heat supply $Q_0\ g(z,t)/\lambda$ is removed. The cooling process's change in temperature, represented by T', fits the differential expression that follows:

$$
\frac{\partial^2 T'}{\partial r^2}+\frac{1}{r}\frac{\partial T'}{\partial r}+\frac{\partial^2 T'}{\partial z^2}=(1+\tau D_{\Omega})\frac{1}{k}\frac{\partial T'}{\partial t}, \tag{4.1}
$$

$$
T'(r,z,t)\big|_{t=t_0}=T(r,z,t_0);, \tag{4.2}
$$

$$
\frac{\partial T'(a,z,t)}{\partial r}=0; \quad t_0\le t, \tag{4.3}
$$

$$
\frac{\partial T'(b,z,t)}{\partial r}=0, \tag{4.4}
$$

$$
T'(r,-h,t)+k_1\frac{\partial T'(r,-h,t)}{\partial z}=0, \tag{4.5}
$$

$$
T'(r,h,t)+k_2\frac{\partial T'(r,h,t)}{\partial z}=0. \tag{4.6}
$$

4.1. The solution of the MDD heat equation in cooling process

Following the same methodology as in the preceding section, the formula for the cooling process's temperature in the Laplace transformation domain is obtained as:

$$
\begin{aligned}
T^{*'}(r,z,s) &= \Bigg\{\frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\frac{[I_0'(qb)k_0(qr)-k_0'(qb)I_0(qr)]}{q[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)}\,\overline{g}^{*}(n,s_0)+\\
&+\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\frac{I_0'(qa)k_0(qr)-k_0'(qa)I_0(qr)}{q[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\,\overline{f}^{*}(n,s_0)\Bigg\}\times\\
&\times\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\frac{I_0'(qa)k_0(qr)-k_0'(qa)I_0(qr)}{q[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\,\overline{f}^{*}(n,s).
\end{aligned}
\tag{4.7}
$$

5. The displacement function's solution

5.1. In heating process

In the case of the Laplace integral domain of the heating process, the displacement function is defined as follows when the value of temperature from calculation Eq.(3.13) is substituted into Eq.(2.8).

$$\begin{aligned}U^{*}(r,z,s)=-(1+\nu)a_t\Bigg\{&\frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\frac{P_n(z)}{q^3\lambda_n}\frac{I_0'(qb)k_0(qr)-k_0'(qb)I_0(qr)}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\bar{g}^{*}(n,s)+\\&+\sum_{n=1}^{\infty}\frac{P_n(z)}{q^3\lambda_n}\frac{I_0'(qa)k_0(qr)-k_0'(qa)I_0(qr)}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\bar{f}^{*}(n,s)\Bigg\}.\end{aligned} \tag{5.1}$$

5.2. In cooling process

Similarly, the displacement function in the Laplace integral domain of the cooling process is written as follows when the temperature from Eq.(4.7) is entered into Eq.(2.8).

$$\begin{aligned}U^{*\prime}(r,z,s)=-(1+\nu)a_t\Bigg\{&\Bigg[\frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\frac{P_n(z)}{q^3\lambda_n}\frac{[I_0'(qb)k_0(qr)-k_0'(qb)I_0(qr)]}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)}\bar{g}^{*}(n,s_0)+\\&+\sum_{n=1}^{\infty}\frac{P_n(z)}{q^3\lambda_n}\frac{I_0'(qa)k_0(qr)-k_0'(qa)I_0(qr)}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\bar{f}^{*}(n,s_0)\Bigg]\times\\&\times\sum_{n=1}^{\infty}\frac{P_n(z)}{q^3\lambda_n}\frac{I_0'(qa)k_0(qr)-k_0'(qa)I_0(qr)}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\bar{f}^{*}(n,s)\Bigg\}.\end{aligned} \tag{5.2}$$

6. Resulting stress components

6.1. In heating process

By replacing Eq.(5.1), we can derive the stress functions for the heating medium in mathematical Eqs (2.10) and (2.11).

$$\begin{aligned}\sigma_{rr}^{*}(r,z,s)=\frac{2\mu(1+\nu)a_t}{r}\Bigg\{&\frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\frac{P_n(z)}{q^2\lambda_n}\frac{I_0'(qb)k_0'(qr)-k_0'(qb)I_0'(qr)}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\bar{g}^{*}(n,s)+\\&+\sum_{n=1}^{\infty}\frac{P_n(z)}{q^2\lambda_n}\frac{I_0'(qa)k_0'(qr)-k_0'(qa)I_0'(qr)}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\bar{f}^{*}(n,s)\Bigg\},\end{aligned} \tag{6.1}$$

$$\begin{aligned}\sigma_{\theta\theta}^{*}(r,z,s)=2\mu(1+\nu)a_t\Bigg\{&\frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\frac{P_n(z)}{q\lambda_n}\frac{I_0'(qb)k_0''(qr)-k_0'(qb)I_0''(qr)}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\bar{g}^{*}(n,s)+\\&+\sum_{n=1}^{\infty}\frac{P_n(z)}{q\lambda_n}\frac{I_0'(qa)k_0''(qr)-k_0'(qa)I_0''(qr)}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\bar{f}^{*}(n,s)\Bigg\}.\end{aligned} \tag{6.2}$$

6.2. In cooling process

The cooling medium's stress functions can be produced by changing Eqs (2.10) and (2.11) to utilize Eq.(5.2) as:

$$\sigma_{rr}^{*\prime}(r,z,s)=\frac{2\mu(1+\nu)a_t}{r}\left\{\left[\frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\frac{P_n(z)}{q^2\lambda_n}\frac{[I_0'(qb)k_0'(qr)-k_0'(qb)I_0'(qr)]}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)}\bar{g}^*(n,s_0)+\right.\right.$$
$$+\left.\sum_{n=1}^{\infty}\frac{P_n(z)}{q^2\lambda_n}\frac{I_0'(qa)k_0'(qr)-k_0'(qa)I_0'(qr)}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\bar{f}^*(n,s_0)\right]\times \tag{6.3}$$
$$\left.\times\sum_{n=1}^{\infty}\frac{P_n(z)}{q^2\lambda_n}\frac{I_0'(qa)k_0'(qr)-k_0'(qa)I_0'(qr)}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\bar{f}^*(n,s)\right\},$$

$$\sigma_{\theta\theta}^{*\prime}(r,z,s)=2\mu(1+\nu)a_t\left\{\left[\frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\frac{P_n(z)}{q\lambda_n}\frac{[I_0'(qb)k_0''(qr)-k_0'(qb)I_0''(qr)]}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)}\bar{g}^*(n,s_0)+\right.\right.$$
$$+\left.\sum_{n=1}^{\infty}\frac{P_n(z)}{q\lambda_n}\frac{I_0'(qa)k_0''(qr)-k_0'(qa)I_0''(qr)}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\bar{f}^*(n,s_0)\right]\times \tag{6.4}$$
$$\left.\times\sum_{n=1}^{\infty}\frac{P_n(z)}{q\lambda_n}\frac{I_0'(qa)k_0''(qr)-k_0'(qa)I_0''(qr)}{[I_0'(qa)k_0'(qb)-I_0'(qb)k_0'(qa)]}\bar{f}^*(n,s)\right\}.$$

7. Special case and numerical computational evaluation

For the purpose of keeping issues easy to understand:

$$f(z,t)=2\delta t(z-h)^2(z+h)^2, \tag{7.1}$$
$$g(z,t)=2(1-e^{-t})(z-h)^2(z+h)^2. \tag{7.2}$$

Applying the Laplace and finite Marchi-Fasulo transforms to expression Eq.(3.17) yields:

$$\bar{f}^*(n,s)=8(k_1+k_2)\left[\frac{(a_nb)\cos^2(a_nb)-\cos(a_nb)\sin(a_nb)}{a_n^2}\right], \tag{7.3}$$

$$\bar{g}^*(n,s)=8(k_1+k_2)\left(\frac{1}{s}-\frac{1}{s+1}\right)\left[\frac{(a_nb)\cos^2(a_nb)-\cos(a_nb)\sin(a_nb)}{a_n^2}\right]. \tag{7.4}$$

For both heating and cooling operations, the distribution of temperatures, displacement processes, and stress components can be easily rewritten by substituting the earlier given values into the resulting formulas.

Aluminium is frequently employed in the aircraft industry and other transportation-related fields because of its low density. In order to execute the numerical computations, we taken into consideration the material features of the metallic aluminium.

Table 1. Thermo-mechanical properties.

Modulus of elasticity	$6.9\times10^{11}\left(dynes / cm^2\right)$
Shear modulus	$2.7\times10^{11}\left(dynes / cm^2\right)$
Poisson ratio	0.281
Thermal diffusivity	$0.86\left(cm^2 / \sec\right)$
Thermal expansion coefficient	$25.5\times10^{-6}\left(cm / cm - {}^0C\right)$
Thermal conductivity	$0.48\left(cal - cm / {}^0C / \sec / cm^2\right)$
Inner radius	$1\,cm$
Outer radius	$3\,cm$

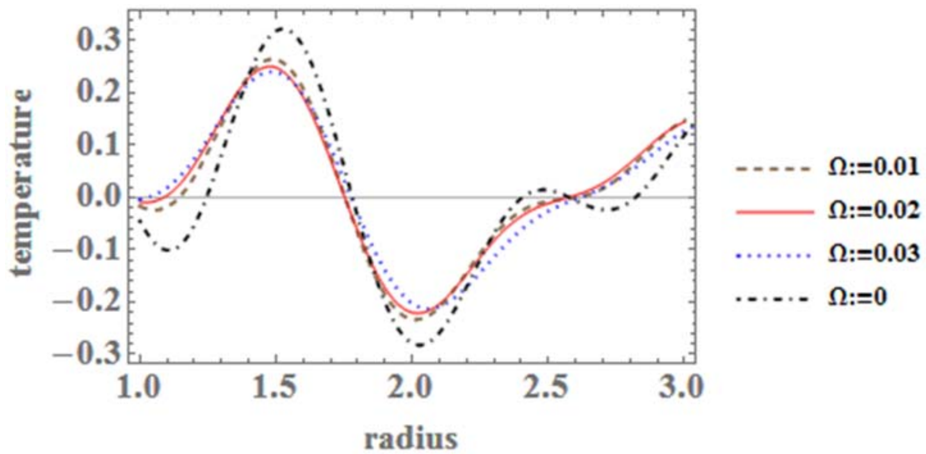

Fig.2(a). Temperature in a radial direction without dimensions while considering the effects of time delay parameters throughout the heating process.

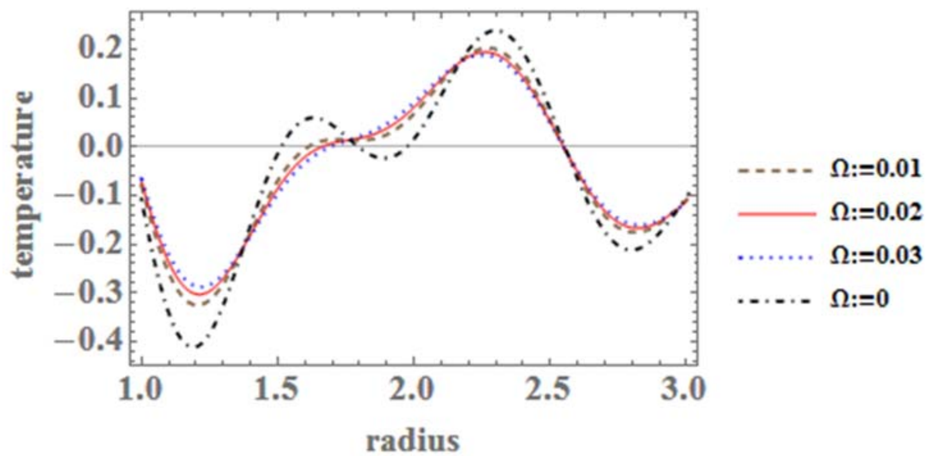

Fig.2(b). Dimensionless temperature in a radial direction under the influence of the cooling process's time delay parameters

This section investigates the effects of the time delay under the heating and cooling processes on the temperature and thermal stress functions along the radial direction. Figures 2 to 4 graphically depict these distributions. It should be mentioned that when the time delay becomes close to infinitesimal and the kernel function is taken to be 1, the current heat transfer model returns to the CV model.

The dimensionless temperature function's fluctuation along the radial direction for various time delay parameters $\Omega = 0, 0.01, 0.02, 0.03$, fixed at $t = 0.5$ and $z = 0.3$, is depicted in Fig.2(a). The temperature rises during the heating process because of the influence of a partially distributed heat source on the inner curving surface. It peaks at $r = 1.5$, then falls until the halfway point and then rises radially outward. A more uniform temperature distribution is observed for longer time delays; alternatively, one may argue that the temperature falls with increasing value of parameter Ω .

The fluctuation of the dimensionless temperature function along the radial direction for various time delay factors $\Omega = 0, 0.01, 0.02, 0.03$, fixed for $t = 0.5$ and $z = 0.3$, is depicted in Fig.2(b). When the partial distributed heat supply at the inner curved surface is removed during the cooling process, the temperature first drops, peaks at $r = 1.25$, and then begins to rise radially outward. In this instance as well, greater time delays are accompanied by a more uniform distribution of temperature that is, a reduction in temperature as parameter Ω 's value rises.

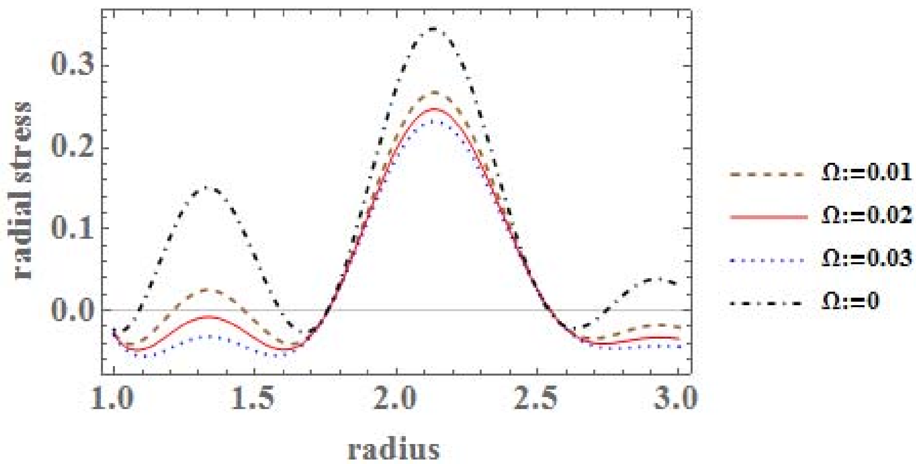

Fig.3(a). Dimensionless radial stress along radial direction under the impact of time delay's parameters in heating process.

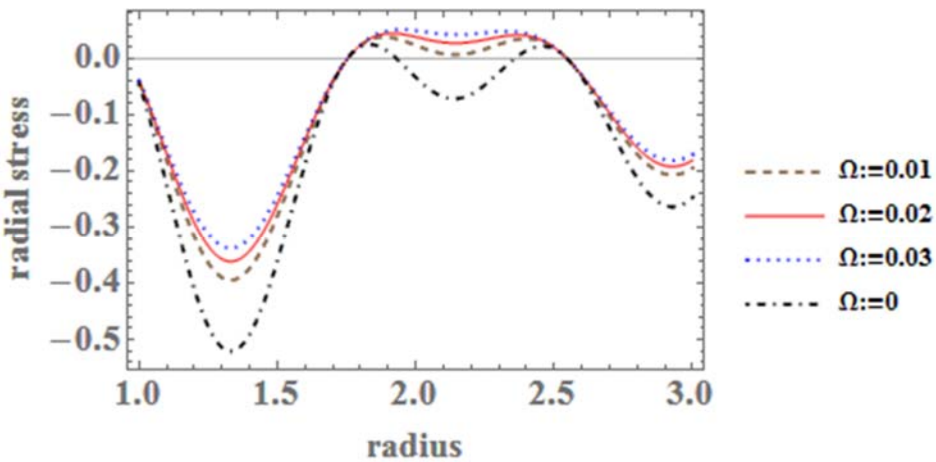

Fig.3(b). Dimensionless radial stress along radial direction under the impact of time delay's parameters in cooling process.

The impact of time delay on the dimensionless radial stress for different parameters $\Omega = 0, 0.01, 0.02, 0.03$ is depicted in Figs. 3(a) and 3(b). $t = 0.5$ and $z = 0.3$ are fixed to record the aforementioned variation. The distribution of radial stress exhibits the characteristics of a sinusoidal wave, reaching its highest amplitude near the halfway of the radial direction. This could be the result of partial heating, whereas the midpoint of the cooling process is where the wave's minimum amplitude is seen, indicating the removal of partial heat source. Plotting shows that time delay characteristics are clearly visible in both the heating and cooling processes. Thus, it can be said that time delays may be essential in characterizing a material's properties and are dependent on the circumstances in which the material's structure is altered by heating and cooling, which can occur in a variety of physical circumstances.

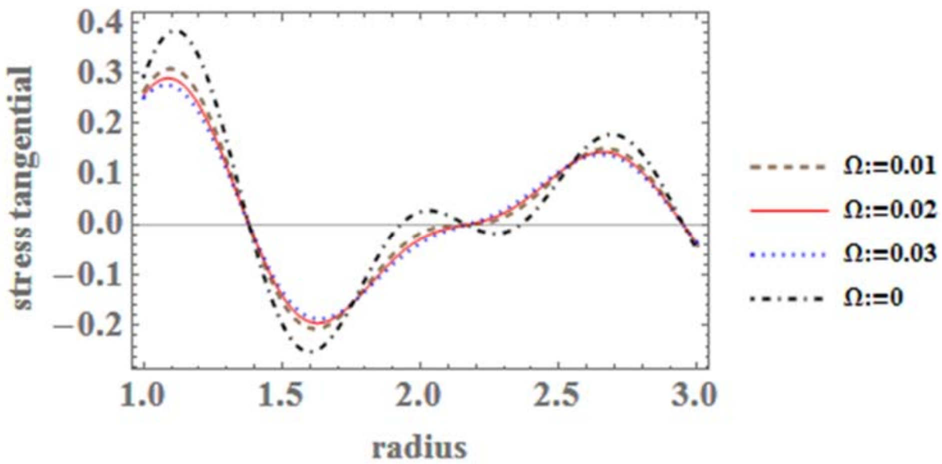

Fig.4(a). Dimensionless tangential stress in a dimensionless radial direction under time delay's influence in heating process.

The substantial impact of time delay on the dimensionless tangential stresses in the heating and cooling processes, respectively, along a dimensionless, radially outward direction is demonstrated by fixing $t = 0.5$ and $z = 0.3$ in Figures 4(a) and 4(b).

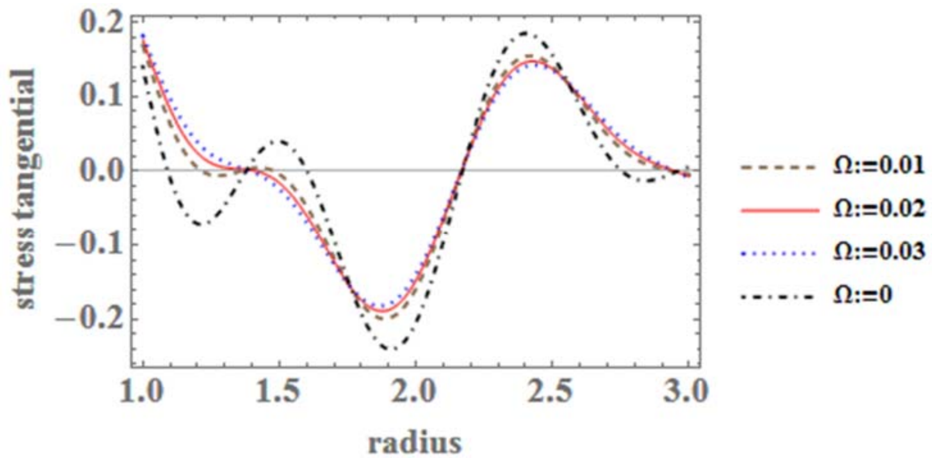

Fig.4(b). Dimensionless tangential stress in a dimensionless radial direction under time delay's influence in cooling process.

At both processes, a small distribution of thermal stress waves is seen at large values of the time delay parameter. Stress first rises during the heating phase after gradually decreasing radially outward and vice versa during the cooling process. Every curve displays tensile stress around the inner and outer radii and compression towards the disk's centre.

8. Conclusion

The current study evaluated the modelling of memory-related heat transfer equations during the heating and cooling process of a thin annular disc. Furthermore, the effects of time delay factors on the temperature distributions, stress functions, and displacements were successfully established. The analytically found mathematical formulae are shown graphically, taking into consideration the material properties of aluminium metal.
The important outcome of the study is highlighted below:

- It is evident from the computed observation that the temperature and stress function variations in the proposed thermoelastic problem are restricted to a limited range.
- Time delays are essential for characterizing a material's properties and depend on the circumstances in which heating and cooling induce structural alterations in the material, which can occur in a variety of physical contexts.
- The temperature, displacement, and stress function fluctuations are significantly impacted by the various values of the time delay parameter.
- Curve variation is a phenomenon that exhibits finite speed propagation.
- It follows that temporal delay, which depend on the conditions under which the material's structure is changed by heating and cooling and can happen in a range of physical circumstances, may be crucial in describing a material's qualities.

Overall, by taking memory-related derivatives under heating and cooling processes into consideration, the study may be helpful for researchers and mathematicians working on the development of thermoelasticity. Surely, this phenomenon helps to clarify a variety of physical processes.

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Nomenclature

a_t – linear coefficient of thermal expansion

$K(t-\xi)$ – kernel function

k – thermal conductivity

n – Marchi-Fasulo transform parameter

$r=a$ – inner radius of disc

$r=b$ – outer radius of disc

s – Laplace transform parameter

T – temperature in heating process

T' – temperature in cooling process

$Q_0\, g(z,t)/\lambda$ – partially source of heat

q – heat flux vector

$z = 2h$ – thickness of disc

ν – Poisson's ratio

μ – Lame's constant

σ_{rr} – radial stress functions

$\sigma_{\theta\theta}$ – tangential stress

Ω – time delay

∇ – spatial gradient operator

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